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Is a Bell-Type Inequality for Nondicotomic Observables a Good Test of Quantum Mechanics?

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Abstract

It is enquired whether a Bell-type inequality for nondicotomic observables established in a preceding paper can be a suitable tool for testing the prediction of quantum mechanics (Q.M.) concerning the correlations $\mathbf{s}_A \cdot \hat{a}\, \mathbf{s}_B \cdot \hat{b}$ observable in a decay of an unpolarized particle of arbitrary spin into two massive particles A and B of spins s_A, s_B, respectively. It is found that such correlations cannot violate the inequality for spins of the subsystems larger than $\frac{1}{2}$ and suggested that, apart from the possibility of finding different sensitive observables, feasible and real tests of QM against local hidden variable theories will probably always imply photon processes, owing to the simplification introduced in the density matrix of the final state by the dicotomic nature of the photon spin variables.

1. *Introduction*

The correlation $\langle \boldsymbol{\sigma}_A \cdot \hat{a}\, \boldsymbol{\sigma}_B \cdot \hat{b} \rangle$ ($\hat{a}$ and $\hat{b}$ are unit vectors in two arbitrary directions) observed in a decay of a spin-0 system with two massive spin-$\frac{1}{2}$

particles A and B as decay products [as in the Einstein, Podolski and Rosen (1935) – EPR – paradox in the version of Bohm and Aharonov (1957)] has been shown by Bell (1964) to satisfy an inequality in a theory of local hidden variables (LHV). Such an inequality has been proved to hold also when the correlations in question are computed as mean values in mixtures of quantum mechanical states of the first kind (Baracca et al., 1975), while it can be violated for suitable choices of the directions if they are computed in full quantum mechanics (QM).[1] Extended to the case of two-photon decays or cascades (either in the singlet or in the triplet $M_z = 0$ state),[2] Bell's inequality has been tested in experiments. The situation is turning in favor of QM against Bell's inequality (Kasday, 1971; Freedman and Clauser, 1972; Faraci et al., 1974; Holt and Pipkin, 1975; Clauser, 1976a, b; Lamehi-Rachti and Mittig, 1975; Fry and Thomson, 1976; Wilson et al., 1976; Bruno et al., 1977), but it is not yet completely clarified, therefore suggesting a broader investigation of checks which may eventually falsify or verify QM in this special aspect.[3]

To this end an extension of Bell's inequality to nondicotomic observables with a discrete and limited spectrum was considered and shown to hold for mixtures of states of the first kind as well as for local hidden variables (Baracca et al., 1976), and an investigation was started in order to verify whether a correlation may be found that could violate it.[4] The results of the above-mentioned paper (Baracca et al., 1976) were as follows:

(a) As far as the correlation $\langle \mathbf{s}_A \cdot \hat{a}\ \mathbf{s}_B \cdot \hat{b} \rangle$ is concerned, it was again computed as a first step for the special case of the decay of a spin-0 system, but now considering s_A and s_B to be arbitrary spins of the massive A and B particles. For simplicity, moreover, $s_A = s_B$ was assumed and orbital angular momentum neglected. The outcome was that in the

[1] This paper is a sequel to an investigation carried out in Baracca et al. (1976), to which we refer for the terminology. We recall that a state of the first kind is one that is a product of states of the constituent systems, and a state that is not a product is said to be of the second kind. In full quantum mechanics decay states, such as the ones we deal with, are described by states of the second kind and give rise, as far as the statistics of the observation on the subsystems is concerned, to mixtures of the second kind, as compared to proper mixtures or mixtures of quantum mechanical states of the first kind.

[2] The extension, although formally very simple, is not trivial, since the dicotomic nature of the photon spin is related to its zero mass. See, e.g. Bohm and Aharonov (1960).

[3] This aspect deals in fact only with the treatment of interacting systems as long as correlations are involved, and it could even not involve at all the other aspects of the theory. It is strictly related – through mixtures of the second kind – also with the "measurement problem", whose solution seems in fact incompatible with the linear character of the Schrödinger equation for interacting systems. See also Baracca et al. (1975).

[4] We recall that dynamical models can be conceived, where spontaneous decays of the pure state of the composite system to a mixture may occur (Baracca et al., 1975; and Bohm and Hiley, 1976); these models would also be tested by Bell-type inequalities. The above spontaneous mechanism seems to be required in the light of the consideration that states of the second kind are requested for the description of *localized* systems (see also footnote 2, page 474, in Baracca et al., 1976).

previous hypothesis such correlation cannot discriminate between QM and the Bell-type inequality for spins larger than $\frac{1}{2}$.

(b) In a completely general case (arbitrary values of the total angular momentum of the decaying systems and of the spin and orbital momentum of the final state) a special set of operators was found that allows one to distinguish between the two cases in any possible case.

A full generalization of these results is needed. In what concerns point (b) the problem is to clarify the most general set of sensitive observables and a real experimental test in the best conditions.

At present, we turn to point (a) and consider in this paper a rather general extension of the investigation carried on in Baracca et al. (1976); namely, we deal with the correlation $P(a, b) = \langle JM | \mathbf{s}_A \cdot \hat{a}\, \mathbf{s}_B \cdot \hat{b} | JM \rangle$ observable in a decay of an unstable particle of arbitrary spin (J, M) into two massive particles A and B of spins s_A, s_B, respectively (s_A being in general different from s_B), taking into account orbital angular momenta. Bell's inequality extended to this case reads (Baracca et al., 1976)

$$|P(a, b) - P(a, c)| \pm [P(a', b) + P(a', c)] \leqslant 2 s_A s_B \tag{1.1}$$

For mixtures of states of the first kind or LHV this inequality has to be satisfied, whereas in QM there might be an appropriate choice of $\hat{a}, \hat{b}, \hat{c}, \hat{a}'$ that leads to a violation. When this is not the case, the observable we are considering does not allow a good test of QM in the sense of Bell's inequality. Computation of the $P(a, b)$'s implies evaluating the mean value of variable $\mathbf{s}_A \cdot \hat{a}\, \mathbf{s}_B \cdot \hat{b}$ in a state $|JM\rangle$ of the two-body decay products, hereafter to be referred to as $|JM\rangle_f$ (f for final); for a given initial state of the decaying system, $|JM\rangle_i$, transitions to $|JM\rangle_f$ states can be classified according to the different values that the channel spin $S(\mathbf{S} = \mathbf{s}_A + \mathbf{s}_B)$ and the relative orbital angular momentum L may assume in the transition, even when parity conservation is requested. The choice of this coupling scheme is first suggested by the fact that the correlation does not involve orbital angular momenta; its practical usefulness arises also from the fact that in general the decay is dominated by the transition in a channel with given L and S: This is, at least, a realistic hypothesis if, e.g., spin orbit or tensor potentials are not very relevant in the final state interaction (FSI) and in general when L–S coupling is good (e.g., SU_4 in nuclear physics or SU_6 in particle physics). The importance of this assumption arises from the factorization it allows of physics and geometry, as will be shown in the next section. In this paper we limit ourselves to considering the decay of an unpolarized system.

Under the above conditions and assumptions it is proved that again, for the operator considered here, Bell's inequality cannot be violated by QM for spins of the subsystems larger than $\frac{1}{2}$. This general result is linked to the fact that the density matrix describing the final state depends on the number of independent parameters which is rapidly increasing with the spins of the subsystems; a larger set of observations than involved by Bell's inequality is therefore needed to determine the density matrix and hence discriminate between QM and LHV.

The effectiveness of the checks based on experiments involving photons is due to the simplification introduced in the density matrix by the dicotomic nature of the photon spin variables. Feasible and real tests of QM will thus probably always imply photon processes. Extensions of the class of photon experiments already considered with this aim, along the lines discussed in this paper, are being considered, together with a more thorough search of different sensitive observables.

2. *The Roles of Dynamics and Geometry in the Evaluation of the $P(a, b)$'s*

Concerning the evaluation of the $P(a, b)$'s, we stress that there is an interplay between dynamics and geometry if the F.S.I. is properly taken into account, the dynamics affecting the weight and phase of the various (L, S) channels. Under the hypothesis discussed in the Introduction, however, the dynamics simply factorizes and drops out, if normalized correlations are duly considered and computed as

$$P(a, b) = \frac{\mathrm{Tr}[\rho s_A \cdot \hat{a}\, s_B \cdot \hat{b}]}{\mathrm{Tr}\,\rho} \tag{2.1}$$

where $\rho = |JM\rangle_{f\,f}\langle JM|$ is the density matrix describing the decay products.

The above hypothesis can in fact be formalized as follows. If the decay process is "weak," and described by the Hamiltonian H_w, one can write

$$|JM\rangle_f = \Omega^{(-)\dagger} H_w |JM\rangle_i \tag{2.2}$$

where

$$\Omega^{(-)} = 1 + \frac{1}{E - H_0 - i\epsilon} V\Omega^{(-)} \tag{2.3}$$

and H_0 is the free Hamiltonian of the initial and final particles and V the final-state "strong" interaction Hamiltonian (Watson, 1952; Gell-Mann and Goldberger, 1953; examples are given in Sakurai, 1964). We can then develop the state $|JM\rangle_f$ into the complete set $|S(s_A, s_B)LJM\rangle$:

$$|JM\rangle_f = \sum_{L,S} |S(s_A, s_B)L\,JM\rangle\langle S(s_A, s_B)L\,JM|\Omega^{(-)}H_w|JM\rangle_i \tag{2.4}$$

Now, if the dynamics is such as to lead to a dominance of the transition in a channel with given L and S, we can write

$$|JM\rangle_f = |S(s_A, s_B)L\,JM\rangle \cdot f_{LS}^{JM} \tag{2.5}$$

where

$$f_{LS}^{JM} = \langle S(s_A, s_B)L\,JM|\Omega^{(-)}H_w|JM\rangle_i \tag{2.6}$$

contains all the physical information. The density matrix for the decay products can then be written

$$\rho = |f_{LS}^{JM}|^2 \cdot \rho_{LS}^{JM} \tag{2.7}$$

with

$$\rho_{LS}^{JM} = |S(s_A, s_B)L\,JM\rangle\langle S(s_A, s_B)L\,JM| \tag{2.8}$$

When (2.7) and (2.8) are substituted into (2.1), the dynamics drops out and we get

$$P(a, b) = \mathrm{Tr}[\rho_{SL}^{JM} \mathrm{s}_A \cdot \hat{a}\, \mathrm{s}_B \cdot \hat{b}] = \langle S(s_A, s_B)L\,JM | \mathrm{s}_A \cdot \hat{a}\, \mathrm{s}_B \cdot \hat{b} | S(s_A, s_B)L\,JM\rangle \tag{2.9}$$

We are thus left with the pure geometrical problem represented by equation (2.9).[5] This feature is an essential requirement for the generality of our calculation, although some specific study of a particular process may certainly reveal the relevance of the interesting interplay between physics and geometry.

3. *General Structure of the Correlation Functions*

It is convenient to work with spherical components. Following the convention[6]

$$s_\pm = \mp\tfrac{1}{2}(s_x \pm i s_y), \qquad s_0 = s_z \tag{3.1}$$

we have

$$\begin{aligned} P(a, b) = \langle S(s_A, s_B)L\,JM | [&-\tfrac{1}{2} s_{A+} s_{B-} (\hat{x}\cdot\hat{a} - i\hat{y}\cdot\hat{a})(\hat{x}\cdot\hat{b} + i\hat{y}\cdot\hat{b}) \\ &- \tfrac{1}{2} s_{A-} s_{B+} (\hat{x}\cdot\hat{a} + i\hat{y}\cdot\hat{a})(\hat{x}\cdot\hat{b} - i\hat{y}\cdot\hat{b}) \\ &+ s_{A_0} s_{B_0} \hat{z}\cdot\hat{a}\,\hat{z}\cdot\hat{b}] | S(s_A, s_A)L\,JM\rangle \end{aligned} \tag{3.2}$$

where we have introduced the basis vectors $\hat{x}, \hat{y}, \hat{z}$; the Cartesian frame may of course be fixed referring to any two of the physical vectors of the problem in a given order, for example, any couple of the directions appearing in (1.1), but is otherwise completely arbitrary. Note that terms such as $s_{A+}s_{B+}$ or $s_{A-}s_{B-}$ do not appear in (3.2): This is due to the fact that the correlation does not involve orbital angular momenta, which causes the third component of the total spin $m_S = m_{s_A} + m_{s_B}$ to be conserved.

The terms in equation (3.2) may be rearranged as follows:

$$\begin{aligned} P(a, b) = \langle S(s_A, s_B)L\,JM | [&-\tfrac{1}{2}(s_{A+}s_{B-} + s_{A-}s_{B+})\hat{a}\cdot\hat{b} \\ &- (i/2)(s_{A+}s_{B-} - s_{A-}s_{B+})(\hat{x}\cdot\hat{a}\hat{y}\cdot\hat{b} - \hat{y}\cdot\hat{a}\hat{x}\cdot\hat{b}) \\ &+ \tfrac{1}{2}(s_{A+}s_{B-} + s_{A-}s_{B+} + 2s_{A_0}s_{B_0})\hat{z}\cdot\hat{a}\hat{z}\cdot\hat{b}] | S(s_A, s_B)L\,JM\rangle \end{aligned} \tag{3.3}$$

[5] One may wonder whether in this problem we still have to deal with states of the second kind. In fact, owing to the dynamical assumption made, such states cannot have anything to do with the composition of S and L. They do, however, arise when coupling of the spins s_A, s_B to give the channel spin S is considered; it can be shown (see the Appendix) that if states of the second kind are replaced by mixtures, a Bell-type inequality can be established.

[6] Hereafter we follow the conventions used by Brink and Satchler (1968).

This expression emphasizes the vector and tensor nature of the second and third term, respectively. Indeed, if the expressions of the products of spherical vectors in (3.3) are solved for the irreducible tensor components, one obtains, in obvious notation,

$$T_0^{(0)} = (1/\sqrt{3})(s_{A+}s_{B-} + s_{A-}s_{B+} - s_{A_0}s_{B_0}) \tag{3.4a}$$

$$T_0^{(2)} = (1/\sqrt{6})(s_{A+}s_{B-} + s_{A-}s_{B+} + 2s_{A_0}s_{B_0}) \tag{3.4b}$$

$$T_0^{(1)} = (1/\sqrt{2}(s_{A+}s_{B-} - s_{A-}s_{B+}) \tag{3.4c}$$

We can then write

$$\begin{aligned} P(a, b) = \langle S(s_A, s_B)LJM \,|\, \{&-\tfrac{1}{2}(1/\sqrt{3})(\sqrt{2}T_0^{(2)} + 2T_0^{(0)})\hat{a}\cdot\hat{b} \\ &- (i/2)\sqrt{2}T_0^{(1)}(\hat{x}\cdot\hat{a}\hat{y}\cdot\hat{b} - \hat{y}\cdot\hat{a}\hat{x}\cdot\hat{b}) \\ &+ (\sqrt{6}/2)T_0^{(2)}\hat{z}\cdot\hat{a}\hat{z}\cdot\hat{b}\} \,|\, S(s_A, s_B)LJM\rangle \end{aligned} \tag{3.5}$$

The matrix element of $T_0^{(1)}$ vanishes because

$$\begin{aligned} \langle S(s_A, s_B) \| T^{(1)} \| S(s_A, s_B)\rangle &= [(2S+1)3(2s_A+1)(2s_B+1)]^{1/2} \\ &\times \begin{Bmatrix} S & S & 1 \\ s_A & s_A & 1 \\ s_B & s_B & 1 \end{Bmatrix} \cdot \langle s_A \| s_A \| s_A\rangle \cdot \langle s_B \| s_B \| s_B\rangle \end{aligned} \tag{3.6}$$

but the 9-j symbol vanishes, as may be seen by interchanging the first two columns and taking into account the fact that $2(S + s_A + s_B)$ is always even.

Hereafter we shall consider the decay of an unpolarized system, which implies averaging over M. Then also the contribution of the matrix element of $T_0^{(2)}$ vanishes, as may be seen by applying the Wigner-Eckart theorem, which reads

$$\langle JM \,|\, T_0^{(2)} \,|\, JM\rangle = \frac{1}{2J+1} \cdot \langle J \| T^{(2)} \| J\rangle \cdot \langle J2M0 \,|\, JM\rangle \tag{3.7}$$

where

$$\langle J2M0 \,|\, JM\rangle = \frac{3M^2 - J(J+1)}{[(2J-1)J(J+1)(2J+3)]^{1/2}} \tag{3.8}$$

When summing over M, $3M^2 - J(J+1)$ averages out to zero.

We are therefore left with the simplified expression

$$\bar{P}(a, b) = \frac{1}{2J+1}\sum_M \langle S(s_A, s_B)LJM \,|\, -\frac{1}{\sqrt{3}}\, T_0^{(0)} \,|\, S(s_A, s_B)LJM\rangle\, \hat{a}\cdot\hat{b} \tag{3.9}$$

Inserting complete sets of the states $|Sm_S LM - m_S\rangle$ and taking into account that $T_0^{(0)}$ does not act on the orbital components, one can rewrite equation (3.9) as

$$\bar{P}(a,b) = \frac{1}{2J+1} \cdot \sum_{M,m_S} |\langle SLJM | Sm_S LM - m_S\rangle|^2 \cdot$$
$$\cdot \left(-\frac{1}{\sqrt{3}}\right) \langle S(s_A, s_B)m_S | T_0^{(0)} | S(s_A, s_B)m_S\rangle \hat{a}\cdot\hat{b} \qquad (3.10)$$

But

$$\sum_{m_S} |\langle SLJM | Sm_S LM - m_S\rangle|^2 = 1 \qquad (3.11)$$

whilst

$$\langle S(s_A, s_B)m_S | T_0^{(0)} | S(s_A, s_B)m_S\rangle$$
$$= \langle S(s_A, s_B) \| T_0^{(0)} \| S(s_A, s_B)\rangle = (1/\sqrt{3})\tfrac{1}{2}[S(S+1) - s_A(s_A+1) - s_B(s_B+1)] \qquad (3.12)$$

which can be thought of as arising from

$$T_0^{(0)} = -(1/\sqrt{3})\mathbf{s}_A \cdot \mathbf{s}_B \qquad (3.13)$$

and

$$\mathbf{s}_A \cdot \mathbf{s}_B = \frac{S(S+1) - s_A(s_A+1) - s_B(s_B+1)}{2} \qquad (3.14)$$

(when the eigenvalues are considered). So that we get finally

$$\bar{P}(a,b) = -\tfrac{1}{6}[S(S+1) - s_A(s_A+1) - s_B(s_B+1)]\hat{a}\cdot\hat{b} \qquad (3.15)$$

In the case of a spin-0 particle decaying into two spin-$\frac{1}{2}$ particles, we have $\bar{P}(a,b) = -\frac{1}{4}\hat{a}\cdot\hat{b}$, a result that can be checked at once by direct computation. For two particles of spin j coupled to zero, $\bar{P}(a,b) = -\frac{1}{3}j(j+1)\hat{a}\cdot\hat{b}$ as found in Baracca et al. (1976).

4. *Bell-Type Inequality*

When expression (3.15) is substituted into Bell's inequality, equation (1.1), one obtains

$$\tfrac{1}{6}|S(S+1) - s_A(s_A+1) - s_B(s_B+1)| \{|\hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{c}| \pm [\hat{a}'\cdot\hat{b} + \hat{a}'\cdot\hat{c}]\} \leqslant 2s_A s_B \qquad (4.1)$$

where the amplitude of the function depending on S, s_A, s_B can be factorized with no loss of generality. It is also easily checked that for integral or half-integral S, s_A, s_B satisfying $|s_A - s_B| \leqslant S \leqslant s_A + s_B$ the above function can never vanish, except for the trivial case $S = s_A = s_B = 0$, as can be physically understood on the basis of the vector model. Equation (4.1) is suitably rewritten as

$$|\hat{a}\cdot(\hat{b}-\hat{c})| \pm [\hat{a}'\cdot(\hat{b}+\hat{c})] \leqslant f(S, s_A, s_B) \qquad (4.2)$$

where we have posed

$$f(S, s_A, s_B) = \frac{12 s_A s_B}{|S(S+1) - s_A(s_A+1) - s_B(s_B+1)|} \qquad (4.3)$$

It is easily checked, as was realized in Baracca et al. (1976), that the left-hand side of equation (4.2) is limited by $-2\sqrt{2}$ and $2\sqrt{2}$, the maximum being reached with the four vectors coplanar, $\hat{a} \perp \hat{a}'$, $\hat{b} \perp c$ and the frame $\hat{a}, \hat{a}'$ rotated $\pi/4$ with respect to the frame $\hat{b}, \hat{c}$.

It shall then be possible to violate Bell's inequality, equation (4.2), for certain configurations of the vectors $\hat{a}, \hat{a}', \hat{b}, \hat{c}$, if the (positive) quantity $f(S, s_A, s_B)$ can turn out to be less than $2\sqrt{2}$ for given values of S, s_A, s_B satisfying the triangular inequalities.

We check at once that, for the case of two $\frac{1}{2}$ spins coupled in a singlet state, $f = 2$; in this special case we thus find again that the inequality can be violated. If, on the other hand, the two spins couple in a triplet state, $f = 6$ and the inequality cannot be violated for whichever choice of the angles.

More generally, we can reason as follows. If $s_A = s_B = j$, $s = 0$

$$f = 6j/(j+1)$$

so that, except for the $\frac{1}{2}$ case already examined, f always exceeds $2\sqrt{2}$. If on the other hand, $s_A \neq s_B$, say $s_A > s_B$, we observe that at fixed s_A and s_B, f shall vary between a minimum and a maximum value while S takes on the values allowed by the triangular inequalities. In particular, the miminum value is attained for the value of S that maximizes the denominator in (4.3), that is $S = s_A + s_B$ when $S(S+1) > s_A(s_A+1) + s_B(s_B+1)$, $S = s_A - s_B$ when $S(S+1) < s_A(s_A+1) + s_B(s_B+1)$. But we have, in the first case, $f = 6$, in the second case $f = 6\, s_A/(s_A+1)$.

We can then conclude that, except for the case already studied, inequality (4.2) can never be violated.

This conclusion can be stated in an expressive way, if the case of large s_A and s_B, and also of large S, is considered. For this case of large quantum numbers, we can put $\langle S(s_A s_B) | \mathbf{s}_A \cdot \mathbf{s}_B | S(s_A s_B)\rangle = s_A s_B \cos(\widehat{\hat{s}_A \hat{s}_B})$. Then, making use of equation (3.15), equation (4.2) can be cast into the form

$$|\cos(\widehat{\hat{s}_A \hat{s}_B})| \cdot \{|\hat{a}\cdot(\hat{b}-\hat{c}) \pm [\hat{a}'\cdot(\hat{b}+\hat{c})]\} \leqslant 6 \qquad (4.4)$$

A violation of the generalized Bell's inequality would then imply

$$\cos(\widehat{\hat{s}_A \hat{s}_B}) > 3/\sqrt{2}$$

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Appendix

We start from

$$P(a,b) = \langle S(s_A, s_B)LJM \,|\, \mathbf{s}_A\cdot\hat{a}\,\mathbf{s}_B\cdot\hat{b} \,|\, S(s_A, s_B)LJM\rangle \tag{A.1}$$

The state $|S(s_A, s_B)LJM\rangle$ can be analyzed in two steps; first, one can write

$$|S(s_A, s_B)LJM\rangle = \sum_{m_S} |Sm_SLM - m_S\rangle\langle Sm_SLM - m_S \,|\, S(s_A, s_B)LJM\rangle \tag{A.2}$$

the states $|Sm_SLM - m_S\rangle$ can be further analyzed as follows:

$$|Sm_SLM - m_S\rangle = |Sm_S\rangle\,|LM - m_S\rangle \tag{A.3}$$

$$|Sm_S\rangle = \sum_{m_A, m_B} |s_Am_As_Bm_B\rangle\langle s_Am_As_Bm_B \,|\, Sm_S\rangle \tag{A.4}$$

subject to the condition

$$m_A + m_B = m_S \tag{A.5}$$

Substitution of (A.3) and (A.4) into (A.2) exhibits the nature of states of the second kind of $|S(s_A, s_B)LJM\rangle$, insofar as it is a linear combination of the direct product states $|s_Am_A\rangle|s_Bm_B\rangle$.

When equations (A.2)-(A.4) are substituted into (A.1), one gets

$$\begin{aligned} P(a,b) = \sum_{m_S}\sum_{m'_S}\sum_{m_A, m'_A}\sum_{m_B, m'_B} &\langle LM - m'_S \,|\, LM - m_S\rangle \\ &\times \langle S(s_A, s_B)LJM \,|\, Sm'_SLM - m'_S\rangle\langle Sm_SLM - m_S \,|\, S(s_A, s_B)LJM\rangle \\ &\times \langle Sm'_SLM - m'_S \,|\, s_Am'_As_Bm'_B\rangle\langle s_Am_As_Bm_B \,|\, Sm_SLM - m_S\rangle \\ &\times \langle s_Am'_A \,|\, \mathbf{s}_A\cdot\hat{a} \,|\, s_Am_A\rangle\langle s_Bm'_B \,|\, \mathbf{s}_B\cdot\hat{b} \,|\, s_Bm_B\rangle \end{aligned} \tag{A.6}$$

where the sums are subject to condition (A.5) and

$$m'_A + m'_B = m'_S = m_S \tag{A.7}$$

Equation (A.6) can be simplified to give

$$\begin{aligned} P(a,b) = \sum_{m_S} &\langle S(s_A, s_B)LJM \,|\, Sm_SLM - m_S\rangle^2 \\ &\times \sum_{m_A, m'_A}\sum_{m_B, m'_B} \langle Sm_SLM - m_S \,|\, s_Am'_As_Bm'_B\rangle \\ &\times \langle s_Am'_As_Bm_B \,|\, Sm_SLM - m_S\rangle\langle s_Am'_A \,|\, \mathbf{s}_A\cdot\hat{a} \,|\, s_Am_A\rangle \\ &\times \langle s_Bm'_B \,|\, \mathbf{s}_B\cdot\hat{b} \,|\, s_Bm_B\rangle \end{aligned} \tag{A.8}$$

With little manipulation, equation (A.8) can be put into a form that can be suitably compared with the form deducible for a mean value of a correlation

(Baracca et al., 1974) from a general theorem by von Neumann. To simplify the task, let us remark that an integral number i can be put into a one-to-one correspondence with the values that m_S assumes varying between $-S$ and S. We further observe that the relations (A.5) and (A.7) determine, at fixed i, m_B (m'_B) in terms of m_A (m'_A); that is, a single integral k_i (k'_i) determines both m_A (m'_A) and $m_B(m'_B)$, i.e., $k_i \leftrightarrow (m_A, m_B)_{m_S}$, $k'_i \leftrightarrow (m'_A, m'_B)_{m_S}$.

We exploit this fact formally by putting

$$\begin{aligned} m_A &= \lambda_{k_i}, \qquad m_B = \rho_{k_i} \\ m'_A &= \lambda_{k'_i}, \qquad m'_B = \rho_{k'_i} \end{aligned} \tag{A.9}$$

and correspondingly, with fairly obvious notation

$$\begin{aligned} |s_A m_A\rangle &= |\varphi_{\lambda_{k_i}}\rangle, \qquad |s_B m_B\rangle = |\xi_{\rho_{k_i}}\rangle \\ |s_A m'_A\rangle &= |\varphi_{\lambda_{k'_i}}\rangle, \qquad |s_B m'_B\rangle = |\xi_{\rho_{k'_i}}\rangle \end{aligned} \tag{A.10}$$

We further write

$$\mathbf{s}_A \cdot \hat{a} = \hat{A}(a), \quad \mathbf{s}_B \cdot \hat{b} = \hat{B}(b) \tag{A.11}$$

where the caret notation stresses the nature of quantum-mechanical operators of $\hat{A}$ and $\hat{B}$.

In this notation, the four sums over m_A, m'_A, m_B, m'_B reduce to two sums over k_i and k'_i. We further observe that the Clebsch-Gordan coefficients within the latter sums are real numbers, depending upon i, k'_i and i, k_i, respectively, which can be written as the power $\frac{1}{2}$ of positive real numbers $w_{k'_i}, w_{k_i}$. In this notation, equation (A.8) can be rewritten as

$$P(a, b) = \sum_i C_i^2 \sum_{k_i, k'_i} w_{k_i}^{1/2} w_{k'_i}^{1/2} \langle \varphi_{\lambda_{k'_i}} | \hat{A}(a) | \varphi_{\lambda_{k_i}} \rangle \langle \xi_{\rho_{k'_i}} | \hat{B}(b) | \xi_{\rho_{k_i}} \rangle \tag{A.12}$$

where the coefficients

$$C_i \equiv \langle S(s_A, s_B) LJM | S m_S L M - m_S \rangle \tag{A.13}$$

are subject to the condition

$$\sum_i C_i^2 = 1 \tag{A.14}$$

Equation (A.12) is strongly reminiscent of equation (2.8) of Baracca et al., (1976), whose right-hand side reads

$$\sum_{k, k'} w_k^{1/2} w_{k'}^{1/2} \langle \varphi_{\lambda_{k'}} | \hat{A}(a) | \varphi_{\lambda_k} \rangle \langle \xi_{\rho_{k'}} | \hat{B}(b) | \xi_{\rho_k} \rangle \tag{A.15}$$

In fact, expression (A.15) proves to hold whenever the subscripts λ and μ are linked by a linear relation of the form

$$f(\lambda, \mu) = 0 \tag{A.16}$$

containing no further parameter. Equation (A.12) exhibits then two interesting features: On the one hand it gives a realization of von Neumann's result less trivial than the usual one discussed in connection with the Bohm–Aharonov version of the EPR paradox; on the other hand it generalizes the result to cases when condition (A.16) is superseded by a slightly more general condition such as (A.5).

Form (A.15) is replaced, in the case of proper mixtures, by the expression

$$P^{\circ}_{(\mathrm{PR})}(a, b) = \sum_k w_k \langle \varphi_{\lambda_k} | \hat{A}(a) | \varphi_{\lambda_k} \rangle \langle \xi_{\rho_k} | \hat{B}(b) | \xi_{\rho_k} \rangle \tag{A.17}$$

where the subscript PR stands for "proper" mixture and the superscript recalls that we are referring to the situation examined previously. It is immediately concluded that, for the same situation, the $P(a, b)$ of equation (A.12) must be replaced by the expression

$$P_{(\mathrm{PR})}(a, b) = \sum_i C_i^2 \sum_{k_i} w_{k_i} \langle \varphi_{\lambda_{k_i}} | \hat{A}(a) | \varphi_{\lambda_{k_i}} \rangle \langle \xi_{\rho_{k_i}} | \hat{B}(b) | \xi_{\rho_{k_i}} \rangle \tag{A.18}$$

It was proved in Baracca et al. (1976) that the combination of $P^{\circ}_{(\mathrm{PR})}(a, b)$ considered by Bell can never exceed Bell's limit. The same combination for $P_{(\mathrm{PR})}(a, b)$ will also satisfy Bell's inequality as a consequence of (A.14).

It is thereby proved that for the situation examined in this paper proper mixtures again cannot violate Bell–type inequalities.

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